Solution of some problems of linear plane theory of elasticity with mixed boundary conditions by the method of boundary integrals

By
Aisha Sleman Melod Gjam

A Thesis Submitted to
Faculty of Science
In partial Fulfillment of the Requirements for
the Degree of M. Sc of Science
(Applied Mathematics)
Mathematics Department
Faculty of Science - Cairo University
2012

Supervisors

Prof. A. F. Ghaleb
Mathematics Department
Faculty of Science
Cairo University

Prof. H. A. Abdusalam
Mathematics Department
Faculty of Science
Cairo University

Dr. Nahed S. Hussien
Mathematics Department
Faculty of Science
Cairo University
APPROVAL SHEET FOR SUBMISSION

Thesis Title: Solution of some problems of linear plane elasticity with mixed boundary conditions by the method of boundary integrals

Name of candidate: Aisha Sleman Melod Gjam

This thesis has been approved for submission by the supervisors:

1- Prof. A. F. Ghaleb
Mathematics Department, Faculty of Science
Cairo University

Signature:

2- Prof. H. A. Abdusalam
Mathematics Department, Faculty of Science
Cairo University

Signature:

3- Dr. Nahed S. Hussien
Mathematics Department, Faculty of Science
Cairo University

Signature:

Prof. N. H. Sweilam

Chairman of Mathematics Department

Faculty of Science - Cairo University
ABSTRACT

Student Name: Aisha Sleman Melod Gjam

Title: Solution of some problems of linear plane theory of elasticity with mixed boundary conditions by the method of boundary integrals.

Degree: M.SC (Applied Mathematics)

A numerical boundary integral scheme is proposed for the solution of the system of field equations of plane, linear elasticity in stresses for homogeneous, isotropic media occupying a simply connected domain under mixed boundary conditions.

The problem is solved in a simply connected region with smooth boundary. The imposed boundary conditions are of two types: (i) type (I) boundary condition, where the force is prescribed on one part of the boundary and (ii) type (II) boundary condition, where the displacement is prescribed on the other part of the boundary. This problem is replaced by two subproblems with homogeneous boundary conditions, one of each type, having a common solution. The equations are reduced to a system of boundary integral equations, which is then discretized in the usual way and the problem at this stage is reduced to the solution of a rectangular linear system of algebraic equations. The unknowns in this system of equations are the boundary values of four harmonic functions which define the full elastic solution, together with the unknown boundary values of stresses or displacements on proper parts of the boundary.
The proposed scheme is first applied to some test problems with known solutions, in order to ensure the efficiency of the method and the correctness of the computing program. Then we solve two problems concerning the circular and the elliptic boundaries, each under a given pressure on one-half of the boundary, and zero displacements on the other half. The obtained results on the boundary and in the bulk are thoroughly discussed. In particular, we put in evidence the singular behavior of a stress component at the two boundary points of separation. A singular solution is considered. Plots on the boundary and three-dimensional plots in the bulk are given.

**Keywords**: Theory of elasticity, plane elasticity, boundary integral method, mixed boundary conditions, collocation method.

**Supervisors:**

1- Prof. Ahmed Fouad Mohamed Fouad Ghaleb

2- Prof. Hosny Ali Abdusalam

3- Dr. Nahed Sayed Mahmoud

Prof. Nasser Hassan Sweilam

Chairman of Mathematics Department
# Table of Contents

Table of Contents iii  
Acknowledgments v  
Introduction vii  

## 1 BASIC PRINCIPLES AND EQUATIONS  
1.1 Problem description ........................................... 1  
1.2 Basic equations .............................................. 2  
1.2.1 Field equations ........................................... 2  
1.2.2 Representation of harmonic functions .................... 4  
1.2.3 Necessary closure conditions ............................. 6  
1.2.4 Boundary conditions ...................................... 6  

## 2 NUMERICAL TREATMENT  
2.1 Notations and primary quantities ............................. 11  
2.2 Discretization procedure ..................................... 15  
2.2.1 Discretization of the harmonic representation ........... 17  
2.2.2 Discretization of the boundary conditions ............... 20  
2.2.3 Discretization of the conditions eliminating the rigid  
body motion .................................................. 25  
2.2.4 Discretization of the additional simplifying conditions 27  
2.2.5 The third fundamental problem ............................ 30  
2.2.6 Solving method .......................................... 31  
2.2.7 Calculation of the harmonic functions at internal points 31  

## 3 TEST PROBLEMS  
3.1 The circular cylinder ......................................... 35  
3.1.1 The first fundamental problem ............................ 36  
3.1.2 Numerical results for a problem with mixed conditions  
on a circular boundary ........................................ 38
Acknowledgments

First of all, gratitude and thanks to ALLAH The Almighty who has always helped and guided me.

Thanks are due to late Prof. M. Z. Abd-Allah, former Head of the Department of Mathematics, Faculty of Science, Cairo University, for his continuous encouragement.

I would like to express my sincere thanks to Prof. A. F. Ghaleb of the Department of Mathematics, Faculty of Science, Cairo University, for suggesting the present point of research and for helpful comments.

I would like to express my deep thanks to Prof. H. A. Abdulsalam of the Department of Mathematics, Faculty of Science, Cairo University, for his continuous encouragement.

I would like to express my deep gratitude to Dr. N. S. Mahmoud of the Department of Mathematics, Faculty of Science, Cairo University, for her guidance during the initial phase of work on this thesis.

Thanks are due to Prof. M. S. Abou-Dina of the Department of Mathematics, Faculty of Science, Cairo University, for his valuable advice and discussions.

I would also like to thank my family for the support they provided me through my entire life, and especially my husband Adel Sleman who stood firmly behind me during the work on this thesis.

Aisha Sleman Melod Gjam, 2012
Introduction

The plane problem of the linear Theory of Elasticity has received considerable attention long ago as being a simplified alternative to the more realistic three-dimensional problems of practical interest. A large class of two-dimensional problems has been tackled using various analytical techniques. Due to the increasing mathematical difficulties encountered in the theoretical studies of problems involving arbitrary boundary shapes or complicated boundary conditions, many purely numerical techniques have been developed in the past decades, which rely on finite difference or finite element techniques. In both methods, the natural boundary of the body is usually replaced by an outer polygonal shape which involves a multitude of corner points and necessarily adds or deletes parts to the region occupied by the body. This, in turn, necessitates the application of boundary conditions on artificial boundaries, a fact that introduces additional inaccuracies into the solution. Minimizing the error requires large computing times. Problems of stability of the numerical scheme are also of crucial importance.

Many of the disadvantages of the numerical techniques are overcome by the use of alternative, semi-analytical treatments based on Boundary Integral Formulations of the problem. Such approaches are usually classified under the general title of Boundary Integral Methods. They have the advantage of producing a formula for the solution and reduce the volume of calculations by considering, at one stage, only the boundary values of the unknown functions and then using them to find the complete solution in the bulk. In addition, the procedure deals exclusively with the real boundary of

The solution of plane problems of elasticity for isotropic media with mixed boundary conditions is a difficult task. Boundary methods may be useful in providing such solutions, especially when the geometry of the domain boundary is not simple. Several papers deal with such problems, either for Laplace’s equation or for the biharmonic equation. Shmegera [21] finds exact solutions of nonstationary contact problems of elastodynamics for a half-plane with friction condition in the contact zone in a closed form. A new method of solution based on the use of Radon transform is used. Schiavone [20] presents integral solutions of mixed problems in plane strain elasticity with microstructure. Haller-Dintelman et al. [9] considers three-dimensional elliptic model problems for heterogeneous media, including mixed boundary conditions. Helsing [10] studies Laplace’s equation under mixed boundary conditions and their solution by an integral equation method. Problems of elasticity are also considered. Lee et al. [15], [14] study singular solutions at corners and cracks in linear elastostatics under mixed boundary conditions. Explicit solutions are obtained. Khuri [13] outlines a general method for finding well-posed boundary value problems for linear equations of mixed elliptic and hyperbolic type, which extends previous techniques. This method is then used to study a particular class of fully
nonlinear mixed type equations.

In joint work, Abou-Dina and Ghaleb [1], [2] proposed a method to deal with the static, plane problems of elasticity in stresses for homogeneous isotropic media occupying simply connected regions. The method relies on the well-known representation of the biharmonic stress function in terms of two harmonic functions and on the integral representation of harmonic functions expressed in real variables. These authors applied their method to a number of examples with boundary conditions of the first, or of the second type only, but did not consider mixed conditions. Constanda [5] discusses Kupradze’s method of approximate solution in linear elasticity. The same author [6] explains the advantages and convenience of the use of real variables due to its generality in dealing with the different forms of the boundary, unlike the approach based on the use of complex variables “where the essential ingredients of the solution must be constructed in full for every individual situation”.

In the thesis, we propose a numerical scheme for the solution of mixed boundary-value problems of plane, linear elasticity for homogeneous, isotropic elastic bodies occupying simply connected domains with smooth boundaries. The imposed boundary conditions are of two types: (i) type (I) boundary condition, where the force is prescribed on one part of the boundary and (ii) type (II) boundary condition, where the displacement is prescribed on the other part of the boundary. More precisely, part of the boundary is subjected to a given pressure, and the remaining part of the boundary is fixed.

This initial problem with mixed boundary conditions is replaced by two subproblems with homogeneous boundary conditions, one of each type, having a common solution. Following the scheme presented in [1], the equations for each of these two subproblems are reduced to a system of boundary integral equations which are then discretized in the usual way, and the problem at this stage is reduced to the solution of a linear system of algebraic equations. The proposed numerical scheme is applied to two problems concerning
the circular and the elliptic boundaries, each under a uniform pressure on one-half of the boundary, and zero displacement on the other half. The obtained results are thoroughly discussed. In particular, we put in evidence the singular behavior of a stress component at the two separation points on the boundary. Boundary graphs and three-dimensional plots for the stress function, the stress components and the displacement components in the whole domain are also given. All figures were produced using Mathematica 7.0 software.

The thesis contains an English Abstract of the thesis, an Arabic Abstract, an Introduction, five Chapters, a summary of Conclusions and a bibliography on the subject of the thesis.

The Introduction contains a brief summary on the subject and the most popular mathematical methods to deal with it.

The first Chapter contains the theoretical principles underlying the plane theory of linear elasticity and the method of boundary integral equations. These theoretical principles are exposed here with some detailed explanations following [1], in order to make the reader familiar with the material of the thesis.

The second Chapter includes the numerical treatment for the method under consideration. The discretization procedure is carried out following [2]. This procedure results in a linear system of algebraic equations whose solution provides, as a first step, the values of the basic two unknown harmonic functions and their harmonic conjugates on the boundary. The elements of the produced matrix for determining these harmonic functions are presented.

The third Chapter involves three test problems with known solutions. We have considered three test problems of elasticity of the first type, i.e. with prescribed stresses on the boundary. For each one of them, we calculated the corresponding displacements and used these data to set up a problem with mixed boundary conditions and known solution. More precisely, the stresses are prescribed on one half of the boundary, while the displacements are prescribed on the other half. In the first section we consider the circular
normal cross-section. The second section deals with the elliptic normal cross-section. The third section treats the nearly-circular normal cross-section. In each case, we apply the proposed method and check for the accuracy of the obtained results. In this way, we have been able to test our method and the computer program used to obtain our numerical results.

The fourth Chapter deals with the circular boundary under a variable pressure with special distribution law on one half of the boundary, while the other half is completely fixed. This problem has no known solution. We put in evidence the singular behavior of one stress component at the two boundary separation points and we consider a singular solution of a special type. Boundary graphs and three-dimensional plots in the bulk are given for the unknown functions. The obtained results indicate the need to introduce domains of possible plastic behavior around the two boundary separation points.

The fifth Chapter deals with the elliptic boundary under similar boundary conditions as for the previous chapter. Here, we include the singular solution discussed earlier to solve the problem. We verify that this addition reduces the errors in the tangential stress component significantly. Boundary graphs and three-dimensional plots in the bulk are also given.

In General Conclusions we have gathered the general conclusions obtained within the thesis.

Two papers including the main results of this thesis have been prepared and sent for publication.

Future work will focus on other types of boundary conditions, and on more general types of the boundary.
Chapter 1

BASIC PRINCIPLES AND EQUATIONS

This Chapter consists of three sections. The first section introduces the geometrical frame of the work. The second section includes the basic equations governing the plane theory of linear elasticity and the method of solution under consideration. The third section embodies the necessary conditions to obtain a complete unique solution. Throughout this chapter, the physical quantities are written in terms of real harmonic functions in a convenient way for later use.

1.1 Problem description

Consider an infinite cylinder from an isotropic, homogeneous, elastic material with a simply connected cross-section $D$ bounded by a contour $C$. We use a Cartesian system of coordinates with origin $O$ inside the domain $D$. The parametric representation of the contour $C$ is:

$$x = x(\theta) \quad \& \quad y = y(\theta),$$  \hspace{1cm} (1.1.1)

where $\theta$ is an angular parameter measured from the $x$-axis. The analysis of [1] is carried out for any parametric representation. In practice, the angular
parameter $\theta$ (the author’s opinion) is the most convenient parameter to use in the present applications.

Let $\tau$ and $\mathbf{n}$ denote the unit vector tangent to $C$ at any arbitrary point, in the positive sense associated with $C$, and the unit outwards normal at this point respectively. One has

$$
\tau = \frac{\dot{x}}{\omega}i + \frac{\dot{y}}{\omega}j \& \mathbf{n} = \frac{\dot{y}}{\omega}i - \frac{\dot{x}}{\omega}j,
$$

(1.1.2)

where the “dot” over a symbol denotes differentiation with respect to the parameter $\theta$, and

$$
\omega = \sqrt{\dot{x}^2 + \dot{y}^2}
$$

(1.1.3)

Clearly, the contour $C$ should belong, at least, to the class $C^1$ so as to uniquely define the unit outwards normal $\mathbf{n}$ at each point.

### 1.2 Basic equations

In this section, the well-known basic equations governing the plan theory of linear elasticity are listed without proof. In conformity with [1], the representation of harmonic functions is briefly discussed.

#### 1.2.1 Field equations

In the absence of body forces, the stress tensor components may be expressed by means of one single auxiliary function, called the stress function or Airy’s function. In fact, the equations of equilibrium are automatically satisfied if the identically non-vanishing “total” stress components are defined through
the stress function $U$ by the relations:

$$
\begin{align*}
\sigma_{xx} &= \frac{\partial^2 U}{\partial y^2}, \\
\sigma_{yy} &= \frac{\partial^2 U}{\partial x^2}, \\
\sigma_{xy} &= \frac{\partial^2 U}{\partial x \partial y}.
\end{align*} 
$$

(1.2.1)

With respect to polar coordinates, the stress components are:

$$
\begin{align*}
\sigma_{rr} &= \frac{1}{r} \frac{\partial U}{\partial r} + \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta^2}, \\
\sigma_{\theta\theta} &= \frac{\partial^2 U}{\partial r^2}, \\
\sigma_{r\theta} &= \frac{1}{r^2} \frac{\partial U}{\partial \theta} - \frac{1}{r} \frac{\partial^2 U}{\partial r \partial \theta}.
\end{align*} 
$$

(1.2.2)

The satisfaction of the compatibility conditions shows that $U$ must satisfy the biharmonic equation. This function may be expressed in terms of two real harmonic functions $\Phi$ and $\Psi$ as follows:

$$
U = x \Phi + y \Phi^c + \Psi, 
$$

(1.2.3)

where the superscript "c" denotes the harmonic conjugate. Now, the stress components are expressed in terms of $\Phi$, $\Phi^c$ and $\Psi$ as:

$$
\begin{align*}
\sigma_{xx} &= x \frac{\partial^2 \Phi}{\partial y^2} + 2 \frac{\partial \Phi}{\partial y} \frac{\partial \Phi^c}{\partial y} + \frac{\partial^2 \Phi^c}{\partial y^2} + \frac{\partial^2 \Psi}{\partial y^2}, \\
\sigma_{yy} &= x \frac{\partial^2 \Phi}{\partial x^2} + 2 \frac{\partial \Phi}{\partial x} \frac{\partial \Phi^c}{\partial x} + \frac{\partial^2 \Phi^c}{\partial x^2} + \frac{\partial^2 \Psi}{\partial x^2}, \\
\sigma_{xy} &= -x \frac{\partial^2 \Phi}{\partial x \partial y} - y \frac{\partial^2 \Phi^c}{\partial x \partial y} - \frac{\partial^2 \Psi}{\partial x \partial y}.
\end{align*} 
$$

(1.2.4)

The generalized Hook’s law reads

$$
\begin{align*}
\sigma_{xx} &= \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \left[ \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial u}{\partial y} \right] + \frac{E}{(1 + \nu)} \frac{\partial u}{\partial x}, \\
\sigma_{xy} &= \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \left[ \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} \right] + \frac{E}{(1 + \nu)} \frac{\partial v}{\partial y}, \\
\sigma_{xy} &= \frac{E}{2(1 + \nu)} \left[ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right].
\end{align*} 
$$

(1.2.5)
Where $E$ and $\nu$ denote Young’s modulus and Poisson’s ratio respectively.

Using the above relations together with (1.2.1), one arrives at:

$$
\frac{E}{1 + \nu} u = -\frac{\partial U}{\partial x} + 4(1 - \nu)\Phi,
$$

$$
\frac{E}{1 + \nu} v = -\frac{\partial U}{\partial y} + 4(1 - \nu)\Phi^c,
$$

which may be rewritten as:

$$
2\mu u = (3 - 4\nu)\Phi - x\frac{\partial \Phi}{\partial x} - y\frac{\partial \Phi}{\partial y} - \frac{\partial \Psi}{\partial x},
$$

$$
2\mu v = (3 - 4\nu)\Phi^c - x\frac{\partial \Phi^c}{\partial y} - y\frac{\partial \Phi^c}{\partial y} - \frac{\partial \Psi}{\partial y}. 
$$

### 1.2.2 Representation of harmonic functions

From Green’s second formula, the value of a function $f$ harmonic in a simply connected region $D$ at an arbitrary field point $(x_0, y_0)$, inside $D$, is given by:

$$
f(x_0, y_0) = \frac{1}{2\pi} \oint_C (f(x,y) \frac{\partial \ln R}{\partial n} - \frac{\partial f(x,y)}{\partial n} \ln R) ds, 
$$

where $R = \sqrt{(x - x_0)^2 + (y - y_0)^2}$, represents the distance between the point $(x_0, y_0)$ inside $D$ and the current integration point $(x, y)$ on the boundary $C$. Integrating by parts and noting that both $f$ and $f^c$ satisfy the Cauchy-Riemann conditions, representation (1.2.8) turns out to have the following equivalent form:

$$
f(x_0, y_0) = \frac{1}{2\pi} \oint_C \left( f(x,y) \frac{\partial \ln R}{\partial n} + f^c(x,y) \frac{\partial \ln R}{\partial \tau} \right) ds. 
$$

The harmonic conjugate of (1.2.9) is

$$
f^c(x_0, y_0) = \frac{1}{2\pi} \oint_C \left( f^c(x,y) \frac{\partial \ln R}{\partial n} - f(x,y) \frac{\partial \ln R}{\partial \tau} \right) ds. 
$$

expressions (1.2.9) and (1.2.10) relate the value of any harmonic function, arbitrary field point $(x_0, y_0)$ inside the region $D$, to the boundary distributions of the function itself and its harmonic conjugate.
When the point \((x_0, y_0)\) tends to a boundary point, relations (1.2.8), (1.2.9) and (1.2.10) yield respectively:

\[
f(x_0, y_0) = \frac{1}{\pi} \oint_C (f(x, y) \frac{\partial \ln R}{\partial n} - \frac{\partial f(x, y)}{\partial n} \ln R) ds, \tag{1.2.11}
\]

\[
f(x_0, y_0) = \frac{1}{\pi} \oint_C (f(x, y) \frac{\partial \ln R}{\partial n} + f^c(x, y) \frac{\partial \ln R}{\partial \tau}) ds, \tag{1.2.12}
\]

\[
f^c(x_0, y_0) = \frac{1}{\pi} \oint_C (f^c(x, y) \frac{\partial \ln R}{\partial n} - f(x, y) \frac{\partial \ln R}{\partial \tau}) ds. \tag{1.2.13}
\]

Expressions (1.2.12) and (1.2.13) are reformulated as:

\[
f(x_0, y_0) = \frac{1}{\pi} \oint_C (f(x, y) \frac{\partial \ln R}{\partial n} + (f^c(x, y) - f^c(x_0, y_0)) \frac{\partial \ln R}{\partial \tau}) ds \tag{1.2.14}
\]

and

\[
f^c(x_0, y_0) = \frac{1}{\pi} \oint_C (f^c(x, y) \frac{\partial \ln R}{\partial n} - (f(x, y) - f(x_0, y_0)) \frac{\partial \ln R}{\partial \tau}) ds, \tag{1.2.15}
\]

with

\[
\frac{\partial \ln R}{\partial \tau} = \nabla (\ln R). \tau = \frac{(x - x_0) \dot{x} + (y - y_0) \dot{y}}{\omega((x - x_0)^2 + (y - y_0)^2)}, \tag{1.2.16}
\]

\[
\frac{\partial \ln R}{\partial \tau} = \nabla (\ln R). n = \frac{(x - x_0) \dot{y} - (y - y_0) \dot{x}}{\omega((x - x_0)^2 + (y - y_0)^2)}. \tag{1.2.17}
\]

The last alternative expressions (1.2.14) and (1.2.15) become obvious if one goes through the following remark:

**Remark 1**

Without any loss of generality, one may write:

\[
\oint_C f(x, y) \frac{\partial \ln R}{\partial \tau} ds = \oint_C (f(x, y) - f(x_0, y_0)) \frac{\partial \ln R}{\partial \tau} ds. \tag{1.2.18}
\]

5
Proof.

\[
\oint_{C} (f(x,y) \frac{\partial \ln R}{\partial \tau}) \, ds = \oint_{C} (f(x,y) - f(x_0, y_0)) \frac{\partial \ln R}{\partial \tau}) \, ds \\
+ \oint_{C} f(x_0, y_0) \frac{\partial \ln R}{\partial \tau} \, ds \\
= \oint_{C} (f(x,y) - f(x_0, y_0)) \frac{\partial \ln R}{\partial \tau}) \, ds \\
+ f(x_0, y_0) \oint_{C} \frac{\partial \ln R}{\partial \tau} \, ds \\
= \oint_{C} (f(x,y) - f(x_0, y_0)) \frac{\partial \ln R}{\partial \tau}) \, ds \\
+ f(x_0, y_0) \ln R)_{C} \\
= \oint_{C} (f(x,y) - f(x_0, y_0)) \frac{\partial \ln R}{\partial \tau}) \, ds.
\]

\[\square\]

1.2.3 Necessary closure conditions

In this section. We discuss the boundary conditions corresponding to the different types of fundamental problems and some additional conditions of physical and mathematical importance, required to close the field equations in order to get a unique solution.

1.2.4 Boundary conditions

The following classification of the mechanical problems is commonly in use in most of the text books. Attention here is focused two types of fundamental problems:

The first fundamental problem of elasticity

In this kind of problems, one has to find the elastic equilibrium of the studied body if the external stresses acting on its boundary are given [2].

Assuming that the density of the given distribution of the total external surface force is \( f = f_x \mathbf{i} + f_y \mathbf{j} = \sigma_{nx} \mathbf{i} + \sigma_{ny} \mathbf{j} \), the boundary conditions take
the form:

\[
\begin{align*}
    f_x &= (x \Phi_{yy} + 2 \Phi_y + y \Phi_{yy} + \Psi_{yy}) \frac{\dot{y}}{\omega} + (x \Phi_{xy} + y \Phi_{xy} + \Psi_{xy}) \frac{\dot{x}}{\omega}, \\
    f_y &= -(x \Phi_{xy} + y \Phi_{xy} + \Psi_{xy}) \frac{\dot{y}}{\omega} - (x \Phi_{xx} + 2 \Phi_x + y \Phi_{xx} + \Psi_{xx}) \frac{\dot{x}}{\omega}.
\end{align*}
\]

(1.2.19)

The second fundamental problem of elasticity

In this kind of problems, one has to find the elastic equilibrium of the studied body if the displacements of the points of its boundary are given [2]. Assuming that the displacement vector is \( \mathbf{d} = d_x \mathbf{i} + d_y \mathbf{j} \), the boundary conditions take the form:

\[
\begin{align*}
    2 \mu d_x &= (3 - 4\nu) \Phi - x \Phi_x - y \Phi_y - \Psi_x, \\
    2 \mu d_y &= (3 - 4\nu) \Phi_y - x \Phi_y - y \Phi_y - \Psi_y.
\end{align*}
\]

(1.2.20)

At this stage, it is noted that the cylinder under consideration is acted upon by either boundary forces or boundary displacements, in such a way that the resulting problem be a plane strain problem.

It is known that the plane problem of elasticity in stresses admits many solutions, including a rigid body motion. In order to remove any ambiguity, one needs to set some additional requirements on the solution. Such conditions were discussed elsewhere (c.f. [1]).

Eliminating the rigid body translation

These conditions are applied only for the first fundamental problem in order to of the rigid body motion in this case. In the case of the second fundamental problem, the absence of the rigid body motion is secured by the nature of the boundary conditions.